On revenue maximization for selling multiple independently distributed items

Xinye Li and Andrew Chi-Chih Yao[1]

Institute for Interdisciplinary Information Sciences, Tsinghua University, Beijing 100084, China

Contributed by Andrew Chi-Chih Yao, May 20, 2013 (sent for review April 27, 2013)

Consider the revenue-maximizing problem in which a single seller wants to sell k different items to a single buyer, who has independently distributed values for the items with additive valuation. The $k=1$ case was completely resolved by Myerson's classical work in 1981, whereas for larger k the problem has been the subject of much research efforts ever since. Recently, Hart and Nisan analyzed two simple mechanisms: selling the items separately, or selling them as a single bundle. They showed that selling separately guarantees at least a $c/\log^2 k$ fraction of the optimal revenue; and for identically distributed items, bundling yields at least a $c/\log k$ fraction of the optimal revenue. In this paper, we prove that selling separately guarantees at least $c/\log k$ fraction of the optimal revenue, whereas for identically distributed items, bundling yields at least a constant fraction of the optimal revenue. These bounds are tight (up to a constant factor), settling the open questions raised by Hart and Nisan. The results are valid for arbitrary probability distributions without restrictions. Our results also have implications on other interesting issues, such as monotonicity and randomization of selling mechanisms.

auction | mechanism design

In the multiple-items auction problem, a seller wants to sell k different items to n bidders who have private values, drawn from some probability distributions, for these items. Economists are interested in studying incentive-compatible mechanisms under which the bidders are incentivized to report their values truthfully. One central question is how to design such mechanisms that can yield the maximal expected revenue for the seller.

The single item case ($k=1$) was resolved by Myerson's classic work (1) for independently distributed item values. The general case of multiple item ($k>1$) is much subtler and not yet completely solved. In recent years, it has been the subject of intensive studies by both economists (e.g., refs. 2–7) and computer scientists (e.g., refs. 8–12). In particular, when the inputs are discrete, much progress has been made on the efficient computation of the optimal mechanism. Another line of investigation is to design simple mechanisms for approximating optimal revenues (13–15) in various situations, such as in the unit-demand setting. For fuller reviews and discussions of the literature, we refer the reader to the papers by Hart and Nisan (16), and Cai et al. (12), and the references therein.

There remain important aspects of the multiple-item auction that are not yet well understood. Most of the known results put restrictions on the distributions (such as refs. 12, 15, and 17). As noted in ref. 16, a precise characterization of optimal mechanisms is still wanting for general distributions, even for simple cases such as for one bidder and two items. The subtlety can be appreciated by considering the following three examples. If the two items are independent and identically distributed over values $\{0,1\}$ uniformly, then selling them separately at price \$1 each yields revenue 1, which is better than bundling (with optimal bundle price \$1 and revenue 3/4). However, if the values are uniform over $\{1,2\}$, then selling them separately (at an optimal price of \$2 each) yields revenue 2, which is worse than bundling them together for price \$3, with revenue $3\cdot 3/4=2.25$. In the third example, let F be the distribution that takes values $\{1,2,4\}$ with probabilities $\{1/6,1/2,1/3\}$. In this case, the unique optimal mechanism (see ref. 7) is to offer the buyer the choice between a 50% lottery for buying any single item for \$1, and buying both surely as a bundle for \$4. (The reader may refer to refs. 3 and 16 for more examples and discussions.)

It is also not known how to characterize the optimal revenue (up to a constant factor) when there is one bidder and k items for large k; it is unsolved even for ER, the equal-revenue distribution as defined in ref. 16 (i.e., when all items have independent cumulative distributions $F(x)=1-1/x$ for $x\geq 1$).

In the hope of addressing the above issues, Hart and Nisan (16) investigated the case of one bidder ($n=1$) and $k>1$. They obtained interesting structural results that are valid for all distributions, and applied them to obtain performance bounds on two natural mechanisms: selling each of the k items separately, and selling all of the k items as a single bundle. Specifically, they showed that selling separately guarantees at least a $c/\log^2 k$ fraction of the optimal revenue; and for identically distributed items, the bundling mechanism yields at least a $c/\log k$ fraction of the optimal revenue.

In this paper, we prove that selling separately guarantees at least $c/\log k$ fraction of the optimal revenue, whereas for identically distributed items, bundling yields at least a constant fraction of the optimal revenue. These bounds are tight (up to a constant factor), settling the open questions raised (16). Our results also have implications on other interesting issues, such as monotonicity and randomization of the selling mechanisms.

It is worth emphasizing that our results are valid for arbitrary distributions without restrictions, in the same spirit as the results of ref. 16. We present a technique called the "core–tails (CT) decomposition" (*3. CT Decomposition and the Core Lemma*) for analyzing the revenue of general distributions, which may be useful for removing the restrictions in previous works such as refs. 12 and 15.

1. Notations and Preliminaries

We follow the notations in ref. 16. A mechanism for selling k items specifies a (possibly randomized) protocol between a seller and a buyer who has a private valuation $x=(x_1,x_2,\ldots,x_k)$ (where $x_i\geq 0$) for the items. The outcome is an allocation specifying the probability $q_i(x)$ of getting each of the k items and an (expected) payment $s=s(x)$ from the buyer to the seller.

The buyer is assumed to act in his self-interest, behaving rationally (paying no more than his value for the goods received), and reporting his valuation $x'=(x'_1,\ldots,x'_k)$ of the k items so as to maximize his utility. Therefore, as is common in economics theory, we consider only mechanisms aligned with these two considerations, so that the buyer is willing to participate and has incentive to report truthfully (that is, $x'=x$, the true valuation of the items). Precisely, we require the mechanism to be individually rational (IR) so that the buyer utility $b(x)=\sum_i x_i q_i(x)-s(x)$ is

Author contributions: X.L. and A.C.Y. designed research; X.L. and A.C.Y. performed research; and A.C.Y. wrote the paper.

The authors declare no conflict of interest.

Freely available online through the PNAS open access option.

[1]To whom correspondence should be addressed. E-mail: andrewcyao@tsinghua.edu.cn.

www.pnas.org/cgi/doi/10.1073/pnas.1309533110

nonnegative for all x, and also incentive compatible (IC) so that for all x, x', $\sum_i x_i q_i(x) - s(x) \geq \sum_i x_i q_i(x') - s(x')$.

For a cumulative distribution $\mathcal{F}$ on $R_+^k (k \geq 1)$, let $REV(\mathcal{F})$ denote the maximal (expected) revenue $E_{x\sim\mathcal{F}}(s(x))$ obtainable by any incentive compatible and individually rational mechanism. We consider two simple mechanisms: (i) selling each item separately, where each item i has a posted price p_i so that the buyer can decide whether or not to take the item; and (ii) selling all of the items as a bundle with a fixed price, so that the buyer gets the whole bundle or nothing. It is easy to see that both mechanisms are IR and IC. Let $SREV(\mathcal{F})$ be the maximal revenue obtainable by selling each item separately, and $BREV(\mathcal{F})$ be the maximal revenue obtainable by selling all of the items as a bundle.

Let $X = (X_1, \ldots, X_k)$ with values in R_+^k distributed according to $\mathcal{F}$. The notation $REV(X)$, $SREV(X)$, $BREV(X)$ will be used interchangeably with $REV(\mathcal{F})$, $SREV(\mathcal{F})$, $BREV(\mathcal{F})$. Note that we have $SREV(X) = REV(X_1) + \cdots + REV(X_k)$ and $BREV(X) = REV(X_1 + \cdots + X_k)$.

In this paper, we consider only independently distributed item values, i.e., $\mathcal{F} = F_1 \times \cdots \times F_k$, where F_i is the cumulative distribution of item i and the F_i values are not necessarily identical. Clearly, $SREV(\mathcal{F}) = \sum_{i=1}^k REV(F_i)$. For a one-dimensional distribution F, Myerson's characterization of the optimal values gives the following:

$$REV(F) = SREV(F) = BREV(F) = \sup_{p\geq 0} p \cdot Pr_{x\sim F}\{X > p\}.$$

For $k > 1$, characterizing the optimal value $REV(\mathcal{F})$ is a subtler issue and has been the subject of extensive studies. Here, we only list some results from ref. 16 that will be needed for our paper.

Theorem 0. [*See Hart and Nisan* (16).] *There exists a constant* $c_0 > 0$ *such that, for all* $k \geq 2$ *and* $\mathcal{F} = F_1 \times F_2 \times \cdots \times F_k$ *where each* F_i *is a one-dimensional distribution*,

$$SREV(\mathcal{F}) \geq \frac{c_0}{(\log_2 k)^2} REV(\mathcal{F});$$

also when all F_i *are identical* ($F_i = F$ *for all* i),

$$BREV(\mathcal{F}) \geq \frac{c_0}{\log_2 k} REV(\mathcal{F}).$$

Hart and Nisan (16) raised the question whether the above two bounds can be replaced by $c/\log_2 k$ and c, respectively. Such bounds would be tight, as the choice of $F_i(x) = 1 - 1/x$ for $x \in [1, \infty)$ $(1 \leq i \leq k)$ achieves these lower bounds. [It was shown in ref. 16 that, for this $\mathcal{F}$, $SREV(\mathcal{F}) = \theta(k)$ and $REV(\mathcal{F}) \geq BREV(\mathcal{F}) = \theta(k \log k)$.] Our main result is to answer this open question affirmatively.

We need the following structural results from ref. 16 as useful tools. First, it is obvious that

$$REV(F_{i_1} \times F_{i_2} \times \cdots \times F_{i_m}) \leq REV(F_1 \times F_2 \times \cdots \times F_k), \quad [1]$$

for any $1 \leq i_1 < i_2 < \cdots < i_m \leq k$.

Lemma A. (*See ref.* 16.) *Let* X *and* Y *be multidimensional random variables. If* X, Y *are independent, then* $REV(X, Y) \leq 2(REV(X) + REV(Y))$.

Notation. Let Z be a k-dimensional random variable. For any (measurable) subset S of R_+^k, let $1_{Z\in S}$ be the indicator random variable that takes on the value 1 if $Z \in S$ and 0 otherwise. We sometimes write $1_{Z\in S_i} Z$ as $1_{S_i} Z$ for brevity when there is no confusion.

Lemma B. (*See ref.* 16.) (*Subdomain Stitching*) *Let* $S_1, S_2, \ldots, S_\ell$ *be* (*measurable*) *subsets of* R_+^k *such that* $\cup_{1\leq i\leq \ell} S_\ell$ *contains the support of* Z. *Then*

$$\sum_{i=1}^{\ell} REV(1_{S_i} Z) \geq REV(Z).$$

Lemma C. (*See ref.* 16.) *For every* $k \geq 2$ *and* $\mathcal{F} = F_1 \times F_2 \times \cdots \times F_k$ *where* $F_i = F$ *are independent and identical distributions, we have* $BREV(\mathcal{F}) \geq \frac{1}{4} SREV(\mathcal{F})$.

2. Main Results

For any one-dimensional distribution F, let $r_F = \sup_{x\geq 0} x(1 - F(x))$. Myersons' classic result says that $REV(F) = r_F$. For a k-dimensional distribution $\mathcal{F}$, we introduce the concept of the core of $\mathcal{F}$ and prove that it plays an essential role in determining $REV(\mathcal{F})$.

Definition: Let $\mathcal{F} = F_1 \times F_2 \times \cdots \times F_k$ be a k-dimensional distribution, where F_i are independent one-dimensional distributions (not necessarily identical). Define the core of $\mathcal{F}$ to be the finite k-dimensional interval as follows:

$$CORE(\mathcal{F}) = [0, kr_{F_1}] \times [0, kr_{F_2}] \times \cdots \times [0, kr_{F_k}].$$

Let $X_{\mathcal{F}}$ be the random variable distributed according to $\mathcal{F}$. Our first result is a structural theorem, identifying $CORE(\mathcal{F})$ as the critical domain that determines how much revenue can be extracted beyond simply selling each item separately.

Remarks: Without loss of generality, we can assume that $r_{F_i} < \infty$ for all i; otherwise, Theorems 1–3 will be trivially true as all revenues will be infinite.

Theorem 1. *There exist constants* $c, c' > 0$ *such that for every integer* $k \geq 2$ *and every* $\mathcal{F} = F_1 \times F_2 \times \cdots \times F_k$,

$$REV(\mathcal{F}) \leq c\ REV\left(1_{CORE(\mathcal{F})} X_{\mathcal{F}}\right) + c'\ SREV(\mathcal{F}).$$

Theorem 1 reduces the original problem dealing with distributions over infinite range into a problem over a finite range, making it possible to use the law of large numbers for our analysis.

Theorem 2. *There exists a constant* $c > 0$ *such that for every integer* $k \geq 2$ *and every* $\mathcal{F} = F_1 \times F_2 \times \cdots \times F_k$,

$$SREV(\mathcal{F}) \geq \frac{c}{\log_2 k} REV(\mathcal{F}).$$

For identically distributed F_i, we show that bundling achieves optimality to within a constant factor.

Theorem 3. *There exists a constant* $c > 0$ *such that for every integer* $k \geq 2$ *and every* $\mathcal{F} = F_1 \times F_2 \times \cdots \times F_k$ *where* $F_i = F$ *are independent and identical distributions*, *we have the following*:

$$BREV(\mathcal{F}) \geq c\ REV(\mathcal{F}).$$

Theorems 2 and 3 answer an open question raised in Hart and Nisan (16); and by the examples given there, these bounds are the best possible.

Theorem 3 also gives insight into the issues of nonmonotonicity and randomization. Hart and Reny (7) observed a counter intuitive phenomenon: *there exist one-dimensional distributions* F_1, F_2 *where* F_2 *stochastically dominates* F_1 (*i.e.*, $Pr\{X_2 > x\} \geq Pr\{X_1 > x\}$ *for all* x), *yet* $REV(F_1 \times F_1) > REV(F_2 \times F_2)$. It raised an interesting open question how large this nonmonotonicity difference can get. Theorem 3 yields as easy corollary that the above anomalous ratio of the revenues is bounded by a constant.

Corollary 1. *There exists a constant* $c > 0$ *such that for any* $k \geq 2$ *and any one-dimensional distributions* F_1, F_2 *where* F_2 *stochastically* *dominates* F_1, *we have* $REV(\mathcal{F}_2) > c\, REV(\mathcal{F}_1)$ *where* $\mathcal{F}_1 = F_1 \times F_1 \times \cdots \times F_1$ *(k times) and* $\mathcal{F}_2 = F_2 \times F_2 \times \cdots \times F_2$ *(k times).*

The corollary is true because *BREV* is monotonic in the sense that $BREV(\mathcal{F}_2) \geq BREV(\mathcal{F}_1)$ if $\mathcal{F}_2$ stochastically dominates $\mathcal{F}_1$. Theorem 3 also implies a constant factor between the revenues of deterministic auctions versus randomized ones. Let $DREV(\mathcal{F})$ denote the maximum revenue derivable by any deterministic *IC* and *IR* mechanism. Noting that bundling is a deterministic mechanism, we have the following.

Corollary 2. *There exists a constant* $c > 0$ *such that for any* $k \geq 2$ *and* $\mathcal{F} = F \times F \times \cdots \times F (k\ times)$ *where F is any one-dimensional distribution,* $DREV(\mathcal{F}) > cREV(\mathcal{F})$.

In the ensuing sections, we will prove Theorems 1 first, followed by Theorems 2 and 3. Theorem 1 provides the basis for focusing attention on $CORE(\mathcal{F})$ only in analyzing mechanisms such as *SREV* and *BREV*, while ignoring contributions involving any tail components. We first introduce this CT decomposition and study its properties in the next section, before proving Theorem 1 in Section 4.

3. CT Decomposition and the Core Lemma

In this section, we discuss a technique called CT decomposition for partitioning a multidimensional distribution into "core" and "tails," such that the optimal revenue can be effectively estimated by focusing only on the core part. This decomposition is key to our analysis of various mechanisms. We let $k \geq 2$.

3.1. CT Decomposition. ***Definition:*** Let $\mathcal{F} = F_1 \times F_2 \times \cdots \times F_k$ be a k-dimensional distribution. For any subset $A \subseteq \{1, 2, \ldots, k\}$, let T_A be defined as $T_A = V_1 \times V_2 \times \cdots \times V_k$ where $V_i = (kr_{F_i}, \infty)$ if $i \in A$, and $V_j = [0, kr_{F_j}]$ if $j \in \overline{A}$.

Thus, $T_\emptyset = CORE(\mathcal{F})$ and the entire region R_+^k is decomposed into 2^k components:

$$R_+^k = \cup_{0 \leq |A| \leq k} T_A = CORE(\mathcal{F}) \cup \left(\cup_{A \neq \emptyset} T_A\right). \quad \textbf{[2]}$$

We are interested in the tail distributions obtained by restricting $\mathcal{F}$ to T_A where $A \neq \emptyset$. In fact, we will only be interested in those nonempty A for which all of its tail sections have positive weights. We give precise definitions in the following.

Definition: Define $p_{F_i} = 1 - F_i(kr_{F_i}) = Pr_{x \sim F_i}\{X > kr_{F_i}\}$. Note that it is always the case that $p_{F_i} \neq 1$; otherwise, the revenue at price kr_{F_i} is positive and equal to kr_{F_i}, exceeding the maximum revenue r_{F_i}, which is a contradiction.

Definition: A subset $A \subseteq \{1, 2, \ldots, k\}$ is said to be proper (relative to $\mathcal{F}$), if $A \neq \emptyset$ and $p_{F_i} > 0$ (and hence $r_{F_i} > 0$) for all $i \in A$. Denote the collection of all proper subsets by $\mathcal{A}$.

We next define formally the tail distribution obtained by restricting $\mathcal{F}$ to T_A, for any proper $A \in \mathcal{A}$. To do so, we first split each one-dimensional distribution F_i at $x = kr_{F_i}$ into two distributions $F_i^{\mathrm{C}}, F_i^{\mathrm{T}}$ as follows.

Definition: Let F_i be a one-dimensional distribution with $0 < r_{F_i} < \infty$ and $p_{F_i} > 0$. Let $F_i^{\mathrm{C}}, F_i^{\mathrm{T}}$ be two distributions obtained from F_i by restricting the random variable $X \sim F_i$ to $(-\infty, kr_{F_i}]$, and to (kr_{F_i}, ∞), respectively, properly normalized. Define

$$F_i^{\mathrm{C}}(x) = \begin{cases} \dfrac{1}{1 - p_{F_i}} F_i(x) & \text{if} \quad x \in (-\infty, kr_{F_i}], \\ 1 & \text{if} \quad x \in (kr_{F_i}, \infty); \end{cases}$$

$$F_i^{\mathrm{T}}(x) = \begin{cases} 0 & \text{if} \quad x \in (-\infty, kr_{F_i}], \\ 1 - \dfrac{1 - F_i(x)}{p_{F_i}} & \text{if} \quad x \in (kr_{F_i}, \infty). \end{cases}$$

Remarks: To simplify the notation, we sometimes abbreviate r_{F_i} as r_i, and p_{F_i} as p_i when there is no confusion. It is also convenient to extend the definition of F_i^{C} to include those F_i for which $p_{F_i} = 0$ by simply letting $F_i^{\mathrm{C}}(x) = F_i(x)$ for all x (but still leaving F_i^{T} undefined) in such cases.

It is easy to check that $F_i^{\mathrm{C}}(x)$ and $F_i^{\mathrm{T}}(x)$ are continuous from the right and monotone nondecreasing in their values ranging from 0 to 1, and thus are indeed valid distributions. We are now ready to define the family of tail distributions of $\mathcal{F}$.

Definition: For any proper subset $A \in \mathcal{A}$, let $A = \{i_1 < i_2 < \cdots < i_m\}$ and $\overline{A} = \{j_1 < j_2 < \cdots < j_{k-m}\}$. Define

$$\mathcal{F}_A^{\mathrm{Tail}} = F_{i_1}^T \times F_{i_2}^T \times \cdots \times F_{i_m}^T,$$

$$\mathcal{F}_{\overline{A}}^{\mathrm{Core}} = F_{j_1}^C \times F_{j_2}^C \times \cdots \times F_{j_{k-m}}^C.$$

Let $\mathcal{F}_A = \mathcal{F}_A^{\mathrm{Tail}} \times \mathcal{F}_{\overline{A}}^{\mathrm{Core}}$ be the resulting distribution defined over region T_A. We refer to the family of distributions $\{\mathcal{F}_A | A \in \mathcal{A}\}$ as the tail distributions of $\mathcal{F}$ induced by the CT decomposition.

Note also that the probability that the value of $X_\mathcal{F}$ falls in the region T_A is equal to $p_A^{\mathrm{Tail}} \times p_{\overline{A}}^{\mathrm{Core}}$ where

$$p_A^{\mathrm{Tail}} = p_{i_1} \times p_{i_2} \times \cdots \times p_{i_m},$$

$$p_{\overline{A}}^{\mathrm{Core}} = (1 - p_{j_1}) \times (1 - p_{j_2}) \times \cdots \times (1 - p_{j_{k-m}}).$$

Finally, we finish this section with some bounds on the basic parameters of F_i^{C} and F_i^{T}.

Lemma 1. $p_{F_i} \leq \frac{1}{k}$ *and* $1 - p_{F_i} \geq 1 - \frac{1}{k}$.

Proof: By definition of r_{F_i}, we have $(kr_{F_i}) \cdot p_F = kr_{F_i}(1 - F_i(kr_{F_i})) \leq r_{F_i}$. □

Lemma 2. $r_{F_i^C} \leq \frac{r_{F_i}}{1 - p_{F_i}}$, *and if* $p_{F_i} > 0$ *then* $r_{F_i^T} \leq \frac{r_{F_i}}{p_{F_i}}$.

Proof:

$$\text{Note that } r_{F_i^C} = \sup_{0 \leq x \leq kr_{F_i}} x\left(1 - F_i^{\mathrm{C}}(x)\right) = \sup_{0 \leq x \leq kr_{F_i}} x\left(1 - \frac{F(x)}{1 - p_F}\right)$$
$$\leq \frac{1}{1 - p_{F_i}} \sup_{0 \leq x \leq kr_{F_i}} x(1 - F_i(x)) \leq \frac{r_{F_i}}{1 - p_{F_i}}.$$

$$\text{If } p_{F_i} > 0 \text{ then } r_{F_i^T} = \sup_{x \geq kr_{F_i}} x\left(1 - F_i^{\mathrm{T}}(x)\right)$$
$$\leq \frac{1}{p_{F_i}} \sup_{x \geq 0} x(1 - F_i(x)) = \frac{r_{F_i}}{p_{F_i}}.$$

□

3.2. The Core Lemma. We now prove a key structural result, which isolates the contributions of the tail regions from that of the core region.

Lemma 3. (Core Lemma)

$$REV(\mathcal{F}) \leq (8e - 7) REV\left(1_{CORE(\mathcal{F})} X_\mathcal{F}\right) + 2 \sum_{A \in \mathcal{A}} p_A^{Tail} \cdot REV(\mathcal{F}_A^{Tail}).$$

Proof: By Subdomain Stitching (Lemma B) and Eq. **2**, we have

$$REV(\mathcal{F}) \leq REV\left(1_{CORE(\mathcal{F})} X_\mathcal{F}\right) + \sum_{1 \leq |A| \leq k} REV(1_{T_A} X_\mathcal{F}) = REV\left(1_{CORE(\mathcal{F})} X_\mathcal{F}\right) + \sum_{A \in \mathcal{A}} REV(1_{T_A} X_\mathcal{F}), \quad \textbf{[3]}$$

where we used the fact that $1_{T_A} X_\mathcal{F} = 0$ if $A \notin \mathcal{A}$. For any $A \in \mathcal{A}$, Lemma A of *1. Notations and Preliminaries* implies the following:

$$REV(1_{T_A}X_{\mathcal{F}}) = p_A^{\text{Tail}} \cdot p_{\bar{A}}^{\text{Core}} \cdot REV\left(\mathcal{F}_A^{\text{Tail}} \times \mathcal{F}_{\bar{A}}^{\text{Core}}\right) \leq p_A^{\text{Tail}} \cdot p_{\bar{A}}^{\text{Core}} \cdot 2\left(REV(\mathcal{F}_A^{\text{Tail}}) + REV(\mathcal{F}_{\bar{A}}^{\text{Core}})\right). \quad [4]$$

Note that, by Eq. **1**,

$$REV(\mathcal{F}_{\bar{A}}^{\text{Core}}) \leq REV(\mathcal{F}_1^{\text{Core}} \times \mathcal{F}_2^{\text{Core}} \times \cdots \times \mathcal{F}_k^{\text{Core}}) = \prod_{\ell=1}^{m} (1-p_\ell)^{-1} REV(1_{CORE(\mathcal{F})}X_{\mathcal{F}}) \leq 4\, REV(1_{CORE(\mathcal{F})}X_{\mathcal{F}}), \quad [5]$$

where we have used Lemma 1 to conclude $\prod_{\ell=1}^{m}(1-p_\ell)^{-1} \leq 4$. We have from Eqs. **3**–**5** that

$$REV(\mathcal{F}) \leq \left(1 + 8\sum_{A\in\mathcal{A}} p_A^{\text{Tail}}\right) REV(1_{CORE(\mathcal{F})}X_{\mathcal{F}}) + 2\sum_{A\in\mathcal{A}} p_A^{\text{Tail}} \cdot REV(\mathcal{F}_A^{\text{Tail}}).$$

As all $p_i \leq 1/k$ by Lemma 1,

$$\sum_{A\in\mathcal{A}} p_A^{\text{Tail}} \leq \sum_{m=1}^{k} \sum_{i_1<\cdots<i_m} p_{i_1}p_{i_2}\cdots p_{i_m} \leq \sum_{m=1}^{k} \binom{k}{m} (1/k)^m \leq \sum_{m=1}^{k} \frac{1}{m!} \leq e-1.$$

We have thus proved Lemma 3. □

4. Proof of Theorem 1

To prove Theorem 1, we will bound the second term on the right-hand side (RHS) of Lemma 3 in terms of $SREV(\mathcal{F})$. By Lemma 2, $r_{F_j^{\text{T}}} \leq \frac{r_{F_j}}{p_j} = \frac{r_j}{p_j}$. By Theorem 0 of *1. Notations and Preliminaries*, we have for any $A \in \mathcal{A}$

$$REV(\mathcal{F}_A^{\text{Tail}}) \leq \frac{(\log_2(m+1))^2}{c_0} SREV(\mathcal{F}_A^{\text{Tail}}) = \frac{(\log_2(m+1))^2}{c_0}\left(r_{F_{i_1}^{\text{T}}} + r_{F_{i_2}^{\text{T}}} + \cdots + r_{F_{im}^{\text{T}}}\right) \leq \frac{(\log_2(m+1))^2}{c_0}\left(\frac{r_{i_1}}{p_{i_1}} + \frac{r_{i_2}}{p_{i_2}} + \cdots + \frac{r_{i_m}}{p_{i_m}}\right).$$

Noting $p_i \leq 1/k$ from Lemma 1, we have

$$\sum_{A\in\mathcal{A}} p_A^{\text{Tail}} \cdot REV(\mathcal{F}_A^{\text{Tail}}) \leq \frac{1}{c_0}\sum_{m=1}^{k} \frac{(\log_2(m+1))^2}{k^{m-1}} \sum_{i_1<\cdots<i_m} (r_{i_1} + r_{i_2} + \cdots + r_{i_m}) = \frac{1}{c_0}\sum_{m=1}^{k} \frac{(\log_2(m+1))^2}{k^{m-1}} \binom{k}{m} \frac{m}{k}(r_1 + r_2 + \cdots + r_k) \leq \frac{1}{c_0}\left(\sum_{m=1}^{k} \frac{(\log_2(m+1))^2}{(m-1)!}\right) SREV(\mathcal{F}). \quad [6]$$

It follows from Eq. **6** and Lemma 3 (the Core Lemma) that

$$REV(\mathcal{F}) \leq c\, REV(1_{CORE(\mathcal{F})}X_{\mathcal{F}}) + c'\, SREV(\mathcal{F}),$$

where $c = 8e - 7$ and $c' = \frac{2}{c_0}\left(\sum_{m\geq 1} \frac{(\log_2(m+1))^2}{(m-1)!}\right)$. This completes the proof of Theorem 1.

5. Proof of Theorem 2

Because of Theorem 1, it suffices to analyze $REV(1_{CORE(\mathcal{F})}X_{\mathcal{F}})$. Note that

$$REV(1_{CORE(\mathcal{F})}X_{\mathcal{F}}) = \prod_{i=1}^{k}(1-p_{F_i}) REV(F_1^C \times \cdots \times F_k^C) \leq \prod_{i=1}^{k}(1-p_{F_i}) \sum_{i=1}^{k} E_{y_i \sim F_i^C}(Y_i).$$

Now, by standard argument,

$$E_{y_i \sim F_i^C}(Y_i) = \int_0^\infty y\, dF_i^C(y) = \int_0^\infty \int_0^y 1\, ds\, dF_i^C(y) = \int_0^\infty \int_s^\infty 1\quad dF_i^C(y)\, ds = \int_0^\infty Pr\{Y_i > s\}\, ds.$$

Note that

$$Pr\{Y_i > s\} = \begin{cases} 1 - \frac{1}{1-p_{F_i}} F_i(s) \leq \frac{1}{1-p_{F_i}}(1 - F_i(s)) & s \in [0, kr_i], \\ 0 & s \in (kr_i, \infty). \end{cases}$$

Therefore,

$$\int_0^\infty Pr\{Y_i > s\} \leq \frac{1}{1-p_{F_i}} \int_0^{kr_i} (1 - F_i(s))ds.$$

The integral on the RHS satisfies the following:

$$\int_0^{kr_i} (1 - F_i(s))\, ds \leq \int_0^{r_i} ds + \int_{r_i}^{kr_i} (1 - F_i(s))\, ds \leq r_i + \int_{r_i}^{kr_i} \frac{r_i}{s}\, ds = (1 + \ln\ k)\, r_i.$$

Thus,

$$REV(1_{CORE(\mathcal{F})}X_{\mathcal{F}}) \leq \prod_{i=1}^{k}(1-p_{F_i}) \sum_{i=1}^{k} E_{y_i \sim F_i^C}(Y_i) \leq \sum_{i=1}^{k}(1 + \ln k) r_i = (1 + \ln k) SREV(\mathcal{F}).$$

By Theorem 1, this implies

$$REV(\mathcal{F}) \leq c\ (1 + \ln k) SREV(\mathcal{F}) + c'\ SREV(\mathcal{F})$$

and hence Theorem 2.

6. Proof of Theorem 3

By assumption, $F_i = F$ and hence $F_i^C = F^C$ for $1 \le i \le k$.

$$REV(1_{CORE(\mathcal{F})}X_{\mathcal{F}}) = \lambda \cdot REV(F_1^C \times \cdots \times F_k^C),$$

$$BREV(1_{CORE(\mathcal{F})}X_{\mathcal{F}}) = \lambda \cdot BREV(F_1^C \times \cdots \times F_k^C),$$

where $\lambda = \prod_{i=1}^{k}(1 - p_{F_i}) \in [1/4, 1/e]$. Let Y be the random variable distributed according to F^C, we consider two cases as follows.

Case 1: $E_{y\sim F^C}(Y) \le 10 r_F$.

Then

$$REV(F_1^C \times \cdots \times F_k^C) \le E_{y_i \sim F_i^C}\left(\sum_{i=1}^{k} Y_i\right) \le 10\, k r_F = 10\, SREV(\mathcal{F}).$$

Theorem 1 implies

$$\begin{aligned} REV(\mathcal{F}) &\le c\lambda \cdot REV(F_1^C \times \cdots \times F_k^C) + c' SREV(\mathcal{F}) \\ &\le (10\, c\lambda + c')SREV(\mathcal{F}) \\ &\le (10\, c + c')SREV(\mathcal{F}). \end{aligned}$$

Because

$$BREV(F_1 \times \cdots \times F_k) \ge \frac{1}{4} SREV(F_1 \times \cdots \times F_k)$$

by Lemma C (from ref. 16; see *1. Notations and Preliminaries*), it follows that

$$BREV(\mathcal{F}) \ge \frac{1}{4} SREV(\mathcal{F}) \ge \frac{1}{4(10c + c')} REV(\mathcal{F}).$$

Case 2: $E_{y\sim F^C}(Y) > 10 r_F$.

$$\begin{aligned} Var(Y) &\le E(Y^2) = \int_0^\infty x^2\, dF^C(x) = \int_0^\infty \int_0^x 2s\, ds\, dF^C(x) \\ &= \int_0^\infty \left(\int_s^\infty dF^C(x)\right) 2s\, ds = \int_0^\infty Pr_{y\sim F^C}\{y \ge s\} 2s\, ds \\ &\le \int_0^{r_F} 2s\, ds + 2\int_{r_F}^{kr_F} \frac{s \cdot r_{F^C}}{s}\, ds \\ &= r_F^2 + r_{F^C} 2(k-1) r_F \\ &\le r_F^2 + 4(k-1) r_F^2 \le 4k\, r_F^2. \end{aligned}$$

Because y_i are all independent, we have

$$Var\left(\sum_{i=1}^{k} Y_i\right) = k\, Var(Y) \le 4k^2 r_F^2,$$

and by Chebycheff's inequality,

$$\begin{aligned} &Pr_{y_i \sim F_i^C}\left\{|Y_1 + \cdots + Y_k - E(Y_1 + \cdots + Y_k)| > \frac{1}{2} E(Y_1 + \cdots + Y_k)\right\} \\ &\le \frac{4k^2 r_F^2}{\left(\frac{1}{2} E(Y)\right)^2} \le \frac{4k^2}{(5k)^2} \le \frac{1}{4}. \end{aligned}$$

It follows that, by selling the k items as a bundle at price $\frac{1}{2}E(Y_1 + \cdots + Y_k)$, we get

$$\begin{aligned} BREV(F_1^C \times \cdots \times F_k^C) &\ge \frac{3}{4} \cdot \frac{1}{2} E(Y_1 + \cdots + Y_k) \\ &\ge \frac{3}{8} REV(F_1^C \times \cdots \times F_k^C). \end{aligned} \quad [7]$$

Also, observe that

$$\begin{aligned} BREV(F_1^C \times \cdots \times F_k^C) &\ge \frac{3}{4} \cdot \frac{1}{2} E(y_1 + \cdots + y_k) \\ &\ge \frac{3}{8} \cdot 10k\, r_F \ge 3 \cdot SREV(\mathcal{F}). \end{aligned} \quad [8]$$

It follows from Theorem 1 and Eqs. **7** and **8** that

$$\begin{aligned} REV(\mathcal{F}) &\le c\, REV(1_{CORE(\mathcal{F})}X_{\mathcal{F}}) + c'\, SREV(\mathcal{F}) \\ &= c\lambda REV(F_1^C \times \cdots \times F_k^C) + c' SREV(\mathcal{F}) \\ &\le \frac{8c\lambda}{3} BREV(F_1^C \times \cdots \times F_k^C) + \frac{c'}{3} BREV(F_1^C \times \cdots \times F_k^C) \\ &= \left(\frac{8c\lambda}{3} + \frac{c'}{3}\right) BREV(F_1^C \times \cdots \times F_k^C) \\ &= \left(\frac{8c}{3} + \frac{c'}{3\lambda}\right) BREV(1_{CORE(\mathcal{F})}X_{\mathcal{F}}) \\ &\le \left(\frac{8c}{3} + \frac{4c'}{3}\right) BREV(\mathcal{F}). \end{aligned}$$

This completes the proof of Theorem 3.

Remarks: As pointed out by a reviewer, the proofs of Theorems 2 and 3 actually show stronger results in special cases: in Theorem 2, if F_i is supported on $[0, kr_i]$, then $REV(F_i) \ge (c/\log k)E(F_i)$; in Theorem 3, if F is supported on $[0, kr]$, then $BREV(F \times \cdots \times F) \ge ckE(F)$.

ACKNOWLEDGMENTS. This work was supported in part by National Natural Science Foundation of China Grant 61033001, the Danish National Research Foundation, and the National Natural Science Foundation of China Grant 61061130540.

1. Myerson RB (1981) Optimal auction design. *Math Oper Res* 6(1):58–73.
2. McAfee R-P, McMillan J (1988) Multidimensional incentive compatibility and mechanism design. *J Econ Theory* 46(2):335–354.
3. Manelli A-M, Vincent D-R (2006) Bundling as an optimal selling mechanism for a multiple-good monopolist. *J Econ Theory* 127(1):1–35.
4. Manelli A-M, Vincent D-R (2007) Multidimensional mechanism design: Revenue maximization and the multiple-good monopoly. *J Econ Theory* 137(1):153–185.
5. Jehiel P, Vehn M-M, Moldovanu B (2007) Mixed bunding auctions. *J Econ Theory* 134(1):494–512.
6. Lev O (2011) A two-dimensional problem of revenue maximization. *J Math Econ* 47(6):718-727.
7. Hart S, Reny P (2012) Maximizing revenue with multiple goods: Nonmonotonicity and other observations. Discussion paper series DP-630 (The Center for the Study of Rationality, Hebrew University, Jerusalem).
8. Hart S, Nisan N (2013) The Menu-size complexity of auctions. *Proceedings of the 14th ACM Conference on Electronic Commerce* (Philadelphia), pp 565–566.
9. Briest P, Chawla S, Hartline J-D, Kleinberg R-D, Weinberg S-M (2010) Pricing randomized allocations. *Proceedings of the 21st ACM-SIAM Symposium on Discrete Algorithms (SODA)* (Austin, TX), pp 585–597.
10. Cai Y, Daskalakis C, Weinberg S-M (2012) An algorithmic characterization of multidimensional mechanisms. *Proceedings of the 44th Annual ACM Symposium on Theory of Computing (STOC)* (New York), pp 459–478.

11. Cai Y, Daskalakis C, Weinberg S-M (2012) Optimal multi-dimensional mechanism design: Reducing revenue to welfare maximization. *Proceedings of the 53rd Annual IEEE Symposium on Foundations of Computer Science (FOCS)* (New Brunswick, NJ), pp 130–139.
12. Cai Y, Daskalakis C, Weinberg S-M (2013) Reducing revenue to welfare maximization: Approximation algorithms and other generalizations. *Proceedings of the 24th ACM-SIAM Symposium on Discrete Algorithms (SODA)* (New Orleans), pp 578–595.
13. Chawla S, Hartline J-D, Kleinberg R-D (2007) Algorithmic pricing via virtual valuations. *Proceedings of the 8th ACM Conference on Electric Commerce (EC)* (San Diego), pp 243–251.
14. Chawla S, Hartline J-D, Malec D-L, Sivan B (2010) Multi-parameter mechanism design and sequential posted pricing. *Proceedings of the 42th Annual ACM Symposium on Theory of Computing (STOC)* (Cambridge, MA), pp 311–320.
15. Cai Y, Huang Z (2013) Simple and nearly optimal multi-item auctions. *Proceedings of the 24th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)* (New Orleans), pp 564–577.
16. Hart S, Nisan N (2012) Approximate revenue maximization with multiple items. *Proceedings of the 13th ACM Conference on Electronic Commerce* (Valencia, Spain), pp 656.
17. Barlow R-E, Marshall A-W, Proschan F (1963) Properties of probability distributions with monotone hazard rate. *Ann Math Stat* 34(2):375–389.

COMPUTER SCIENCES

ECONOMIC SCIENCES